

Massive spin-2 particles via embedment of the Fierz-Pauli equations of motion

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Abstract

Here we obtain alternative descriptions of massive spin-2 particles by an embedding procedure of the Fierz-Pauli equations of motion. All models are free of ghosts at quadratic level although most of them are of higher order in derivatives. The models that we obtain can be nonlinearly completed in terms of a dynamic and a fixed metric. They include some $f(R)$ massive gravities recently considered in the literature. In some cases there is an infrared (no derivative) modification of the Fierz-Pauli mass term altogether with higher order terms in derivatives. The analytic structure of the propagator of the corresponding free theories is not affected by the extra terms in the action as compared to the usual second order Fierz-Pauli theory.

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1 Introduction

Massive gravity has become an area of intense work in last 5 years, see the review works [1, 2] and references therein. Part of the motivation has an earlier origin, they come from the experimental data of the supernova team [3, 4], which indicate an accelerated expansion of the universe at large distances. A tiny mass for the graviton would certainly diminish the gravitational interaction at large distances which might avoid the introduction of dark energy. From the theoretical standpoint the motivation comes from the new formulations of massive gravity of [5, 6], see also [7, 8], which overcome two important obstacles. Namely, the Boulware-Deser ghost [9] and the van Dam-Veltman-Zakharov mass discontinuity [10, 11] which is solved along the lines of [12].

The theories [6, 7] are built up on the top of the second order (in derivatives) paradigmatic Fierz-Pauli (FP) theory [13]. Even more recent works [14, 15, 16] searching for alternative kinetic terms for massive gravity, reduce to the FP model at linearized level. On the other hand, in order to accommodate a larger class of stable cosmological solutions, higher order modifications of [7] and [6] have been recently considered [17, 18, 19, 20, 21]. So it is natural to ask for possible higher order alternatives to the FP paradigm.

Here we start with the second order FP theory and add quadratic terms in its equations of motion and its derivatives as a technique to generate higher order dual theories which have the same particle content of the massive FP model at quadratic level. Namely, a massive spin-2 particle (in $D = 4$) and no ghosts. By requiring that the FP equations of motion are not only embedded but in fact follow from the higher order model, we end up with the condition (26) which has several solutions. In the next section we write down explicitly four families of solutions. It turns out that we can always find a nonlinear completion of those quadratic (free) theories in terms of a dynamic $g_{\mu\nu}$ and a fixed $g_{\mu\nu}^{(0)}$ metric. Those theories might be considered as alternative starting points for the addition of possible extra terms required for the absence of ghosts at nonlinear level. In particular, a subset of the models (20) have been already reformulated as ghost free theories in [19, 21] by adding nonlinear non derivative terms as in [5, 6]. In section 3 we reinterpret our results in terms of the analytic structure of the propagator while in section 4 we explain the essential differences of our embedment and the Noether gauge embedment where the analytic structure of the propagator is modified. In section 5 we draw our conclusions.

2 Embedding the Euler tensor

Although we use in this work mostly a symmetric tensor $h_{\mu\nu} = h_{\nu\mu}$, it is convenient for the introduction of the embedding idea to recall some results of the works [22, 23] where a generic nonsymmetric tensor $e_{\mu\nu} = h_{\mu\nu} + B_{\mu\nu}$, with $B_{\mu\nu} = -B_{\nu\mu}$, is employed. The nonsymmetric FP model (NSFP) is given by¹

¹Throughout this work we use $\eta_{\mu\nu} = (-, +, \dots, +)$.

$$\begin{aligned}
\mathcal{L}_{NSFP} &= \mathcal{L}_{FP}[h_{\mu\nu}] + \frac{m^2}{2} B_{\mu\nu}^2 \\
&= \frac{1}{2} h^{\mu\nu} (\Box - m^2) h_{\mu\nu} - \frac{1}{2} h (\Box - m^2) h + (\partial^\mu h_{\mu\nu})^2 - \partial^\mu h \partial^\nu h_{\mu\nu} + \frac{m^2}{2} B_{\mu\nu}^2
\end{aligned} \tag{1}$$

We have concluded in [22] that besides the NSFP theory, there are two extra one-parameter families of models in $D = 4$ describing massive spin-2 particles via a nonsymmetric tensor, namely²,

$$\mathcal{L}(a_1) = \mathcal{L}_{NSFP} + \left(a_1 - \frac{1}{4}\right) (\partial^\mu h_{\mu\nu} + \partial^\mu B_{\mu\nu} - \partial_\nu h)^2 = \mathcal{L}_{NSFP} + \left(a_1 - \frac{1}{4}\right) \left(\frac{\partial^\mu \tilde{K}_{\mu\nu}}{m^2}\right)^2, \tag{2}$$

and

$$\begin{aligned}
\mathcal{L}(c) &= \mathcal{L}_{NSFP} - \frac{1}{3} (\partial^\mu h_{\mu\nu} + \partial^\mu B_{\mu\nu} - \partial_\nu h)^2 - m^2 \frac{(1+c)}{2} h^2 \\
&= \mathcal{L}_{NSFP} - \frac{1}{3m^4} \left(\partial^\mu \tilde{K}_{\mu\nu}\right)^2 - \frac{(1+c)}{18m^2} \left[\tilde{K} + \frac{2}{m^2} \partial^\mu \partial^\nu \tilde{K}_{\mu\nu}\right]^2
\end{aligned} \tag{3}$$

where a_1 and c are arbitrary real constants and $\tilde{K} = \eta_{\mu\nu} \tilde{K}^{\mu\nu}$ is the trace of the Euler-tensor $\tilde{K}^{\mu\nu}$ of the NSFP theory which can be written in terms of the Euler tensor $K^{\mu\nu}$ of the usual (symmetric) Fierz-Pauli theory :

$$\tilde{K}^{\mu\nu} \equiv \frac{\delta S_{NSFP}}{\delta e_{\mu\nu}} = K^{\mu\nu} + m^2 B^{\mu\nu}, \tag{4}$$

with

$$\begin{aligned}
K^{\mu\nu} &\equiv \frac{\delta S_{FP}}{\delta e_{\mu\nu}} \\
&= (\Box - m^2) h^{\mu\nu} + \partial^\mu \partial^\nu h - \partial^\mu \partial_\alpha h^{\alpha\nu} - \partial^\nu \partial_\alpha h^{\alpha\mu} + \eta^{\mu\nu} (\partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h + m^2 h)
\end{aligned} \tag{5}$$

Although the antisymmetric field $B_{\mu\nu}$ is completely decoupled in (1), its presence allows us to figure out that the families $\mathcal{L}(a_1)$ and $\mathcal{L}(c)$ differ from the NSFP theory by the addition of quadratic terms in its equations of motion, see (2) and (3). Such feature automatically guarantees that the NSFP equations of motion $\tilde{K}^{\mu\nu} = 0$ minimize also the actions $S(a_1)$ and $S(c)$. However, in order to make sure that the addition of the square terms does not change the particle content of the original theory, we must go beyond the embedding and check that the new equations of motion are completely equivalent to the original ones. In general, the naive addition of square terms in the equations of motion of some given theory will lead to a physically different model.

²The model $\mathcal{L}(a_1)$ at $a_1 = -1/4$ has appeared before in [24]

In what follows, motivated by (2) and (3), we use the addition of quadratic terms in the Euler tensor as a technique to generalize the usual symmetric ($e_{\mu\nu} = e_{\nu\mu} = h_{\mu\nu}$) massive Fierz-Pauli theory. We start with the general Ansatz :

$$\mathcal{L}_G[h_{\mu\nu}] = \mathcal{L}_{FP}[h_{\mu\nu}] + \frac{a}{2} (\partial_\mu K^{\mu\nu})^2 + b \partial_\mu K \partial_\alpha K^{\alpha\mu} + \frac{c}{2} (\partial_\mu \partial_\nu K^{\mu\nu})^2 + \frac{d}{2} K_{\mu\nu}^2 + \frac{f}{2} K^2 \quad (6)$$

where $K^{\mu\nu}$ is given in (5) and \mathcal{L}_{FP} is defined in the two lines of (1). The coefficients $a = a(\square)$, $b = b(\square)$, $c = c(\square)$, $d = d(\square)$ and $f = f(\square)$ are in principle arbitrary analytic functions (Taylor series) of $\square = \partial^\mu \partial_\mu$. The equations of motion of the general Ansatz can be cast in the form

$$K_G^{\mu\nu} = \frac{\delta S_G}{\delta h_{\mu\nu}} = K^{\mu\nu} + \hat{a}^{\mu\nu}_{\alpha\beta} K^{\alpha\beta} = 0 \quad . \quad (7)$$

where \hat{a} is a differential operator given in terms of the arbitrary coefficients of the Ansatz (6). Notice that the FP equations of motion $K^{\mu\nu} = 0$ are embedded in the set of solutions of (7). By applying \hat{a} on (7) we obtain (suppressing indices)

$$\hat{a} K + \hat{a}^2 K = 0 \quad . \quad (8)$$

Our task is to fix the free coefficients in (6) such that $\hat{a}^2 K = 0$. Consecutively, we have on shell $\hat{a} K = 0$, back in (7) we deduce $K^{\mu\nu} = 0$. So we prove the equivalence of (7) and the original equations of motion of the massive Fierz-Pauli theory. Splitting according to the tensor structure, we can write down explicitly

$$\begin{aligned} (\hat{a}^2 K)^{\mu\nu} &= d^2 (\square - m^2)^2 K^{\mu\nu} + A_5 [2d(\square - m^2) + A_5 \square] (\partial^\mu \partial_\alpha K^{\alpha\nu} + \partial^\nu \partial_\alpha K^{\alpha\mu}) \\ &+ \eta^{\mu\nu} [j_1(\square) R_L + j_2(\square) h + \dots] + \partial^\mu \partial^\nu [g_1(\square) R_L + g_2(\square) h + \dots] \quad . \end{aligned} \quad (9)$$

where $A_5 = m^2 a(\square)/2 - d(\square)$ while $R_L = \partial^\mu \partial^\nu h_{\mu\nu} - \square h$ may be interpreted as a linearized scalar curvature about a flat background ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$). The dots stand for terms which vanish³ at $A_5 = 0 = d$. We keep the first line of (9) for later reference. The functions $j_i(\square)$, $g_i(\square)$, $i = 1, 2$ are given by

$$j_1(\square) = -m^2 A_1 (A_1 \square + A_2 \square^2) - m^2 A_3 (D A_1 + A_2 \square) + m^2 \frac{(D-2)}{(D-1)} j_2(\square) \quad (10)$$

$$j_2(\square) = m^2 (D-1) [A_1 (A_3 \square + A_4 \square^2) + A_3 (D A_3 + A_4 \square)] \quad (11)$$

$$g_1(\square) = -m^2 A_2 (A_1 \square + A_2 \square^2) - m^2 A_4 (D A_1 + A_2 \square) + m^2 \frac{(D-2)}{(D-1)} g_2(\square) \quad (12)$$

$$g_2(\square) = m^2 (D-1) [A_2 (A_3 \square + A_4 \square^2) + A_4 (D A_3 + A_4 \square)] \quad (13)$$

where

$$A_1 = [b(D-2) + c m^2] \square - m^2 b (D-1) \quad ; \quad A_2 = -b(D-2) - c m^2 \quad , \quad (14)$$

$$A_3 = -[b m^2 + f(D-2)] \square + f(D-1) m^2 \quad ; \quad A_4 = b m^2 + f(D-2) \quad . \quad (15)$$

³The coefficients a, b, c, d, f are always supposed to be functions of \square unless otherwise stated.

Regarding the tensor structure, the only terms proportional to $h^{\mu\nu}$ and $\partial^\mu \partial_\alpha h^{\alpha\nu} + \partial^\nu \partial_\alpha h^{\alpha\mu}$ in (9) come respectively from $K^{\mu\nu}$ and $\partial^\mu \partial_\alpha K^{\alpha\nu} + \partial^\nu \partial_\alpha K^{\alpha\mu}$. Thus, in order that $(\hat{a}^2 K)^{\mu\nu} = 0$ we must have $d = 0 = A_5$ which is equivalent to $d = 0 = a$. The remaining terms in $(\hat{a}^2 K)^{\mu\nu}$ will vanish **identically** if we require $j_1 = 0 = j_2 = g_1 = g_2$. The reader can check that there are only two solutions to those equations according to $f = 0$ or $f \neq 0$. We have respectively,

$$f = 0 = b \quad , \quad (\text{Solution I}) \quad (16)$$

$$c = \frac{b^2}{f} \quad ; \quad b = \frac{f [m^2 D - (D-2)\Box]}{2 m^2 \Box} \quad , \quad (\text{Solution II}) \quad (17)$$

In solution I, the coefficient $c(\Box)$ remains an arbitrary analytic function. In solution II the arbitrariness lies in the coefficient $f(\Box)$ which can be any Taylor series which must start however, at the second power \Box^2 by locality reasons as we will see later.

In the case of solution I the Ansatz (6) becomes :

$$\mathcal{L}^I = \mathcal{L}_{FP}[h_{\mu\nu}] + \frac{1}{2} (\partial^\mu \partial^\nu h_{\mu\nu} - \Box h) c(\Box) (\partial^\mu \partial^\nu h_{\mu\nu} - \Box h) \quad . \quad (18)$$

The function $c(\Box)$ must have inverse mass squared dimension. The equations of motion $\delta S^I = 0$ are given by

$$K^{\mu\nu} + c(\Box) \Box \theta^{\mu\nu} \Box \theta^{\alpha\beta} h_{\alpha\beta} = 0 \quad . \quad (19)$$

where $\Box \theta^{\mu\nu} = \eta^{\mu\nu} \Box - \partial^\mu \partial^\nu$. Instead of using the operator \hat{a} , defined in (7), it is simpler to apply $\partial_\mu \partial_\nu$ on (19). Only the mass terms of (5) will contribute and we get $\Box \theta^{\mu\nu} h_{\mu\nu} = 0$, back in (19) we recover the usual FP equations of motion $K^{\mu\nu} = 0$. So, although the higher order term in (18) is not a total derivative and contains more than two time derivatives, the new theory \mathcal{L}^I is on shell equivalent to the usual second order FP theory.

Remarkably, the theory \mathcal{L}^I has nonlinear completions. We can choose for instance

$$S^I[g_{\mu\nu}, g_{\mu\nu}^{(0)}] = \frac{1}{2 \kappa^2} \int d^D x \sqrt{-g} \left[R + R c(\Box) R - \frac{m^2}{4} (h_{\mu\nu} h^{\mu\nu} - h^2) \right] \quad . \quad (20)$$

where $h_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}^{(0)}$ while $g_{\mu\nu}^{(0)}$ is some fixed metric as opposed to the dynamic one $g_{\mu\nu}$. In the last term of (20) the indices of $h_{\mu\nu}$ are raised with $g_{\mu\nu}$.

Expanding S^I about flat space $g_{\mu\nu}^{(0)} = \eta_{\mu\nu}$ with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we recover (18) at quadratic order. The constant κ has mass dimension $1 - D/2$.

The fixed metric breaks the general coordinate invariance of the scalar curvature terms but it could be restored by introducing Stückelberg fields in different ways, as explained in [1] in the $c = 0$ case.

According to [21] the addition of fine tuned, as in [6], higher powers of $h_{\mu\nu}$ (without derivatives) may render the model (20), with constant $c(\Box) = c_0$, ghost free at nonlinear level. In [21] the cosmology of the $R + c_0 R^2$ model, and of an $f(R)$ generalization thereof, has been investigated leading to both early time inflation and late time accelerated expansion.

The solution II corresponds to

$$\mathcal{L}^{II} = \mathcal{L}_{FP}[h_{\mu\nu}] + \frac{\Phi f(\square) \Phi}{8\square^2} \quad . \quad (21)$$

where

$$\Phi = [(D-2)\square + Dm^2] R_L + 2m^2(D-1)\square h \quad . \quad (22)$$

The analytic function $f(\square)$ is arbitrary except for the fact that it must start at the second power \square^2 as required by locality.

The contribution of the last term of (21) to the equations of motion $\delta S^{II}/\delta h_{\mu\nu} = 0$ is proportional to the scalar Φ . If we show that $\Phi = 0$ on shell, then we prove the equivalence between $\delta S^{II}/\delta h_{\mu\nu} = 0$ and $K^{\mu\nu} = 0$. In order to show that $\Phi = 0$ we can take a general combination $s(\square)\partial_\mu\partial_\nu + t(\square)\eta_{\mu\nu}$ and apply on $\delta S^{II}/\delta h_{\mu\nu} = 0$. It turns out that if we choose $s(\square) = Dm^2 - (D-2)\square$ and $t(\square) = -2m^2\square$ we derive $\Phi = 0$. Consequently, once again, though the last term in (21) is not a total derivative, the equations of motion of (21) are equivalent to the usual FP equations $K^{\mu\nu} = 0$.

The theory (21) also has nonlinear completions, for instance,

$$S^{II}[g_{\mu\nu}, g_{\mu\nu}^{(0)}] = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} \left[R + \frac{\Phi \tilde{f}(\square) \Phi}{2} - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2) \right] \quad . \quad (23)$$

where $\tilde{f}(\square) = f(\square)/(8\square^2)$ is an arbitrary Taylor series in $\square = \nabla_\mu\nabla^\mu$ and Φ is defined as in (22) with the replacement, for instance, of $R_L(h)$ by the full scalar curvature $R(g)$ and $\square = \nabla_\mu\nabla^\mu$. Moreover, $h_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha\beta}^{(0)}$ is a rank-2 tensor under general coordinate transformations. We assume that $g_{\alpha\beta}^{(0)}$ has been “covariantized” into a rank-2 tensor under general coordinate transformations via introduction of Stückelberg fields, as explained, e.g., in [1]. The operator $\square = \nabla_\mu\nabla^\mu$ is a contraction of two covariant derivatives with respect to the metric $g_{\mu\nu}$. Expanding S^{II} about flat space we recover (21).

We have obtained solutions I and II from the condition $\hat{a}^2 K = 0$. The first line of (9) vanished by imposing $a = 0 = d$ while the the second line vanished **identically** by requiring $j_1 = 0 = j_2 = g_1 = g_2$. However, we could have instead a weaker condition where $\hat{a}^2 K = 0$ holds only on shell, i.e., as a consequence of $K_G^{\mu\nu} = 0$. One can convince oneself that the first line of (9) can not vanish on shell, so we still need to fix $a = 0 = d$. Regarding the second line, if we had $\partial^\mu\partial^\nu h_{\mu\nu} = 0 = h$ as a consequence of the full equations of motion $K_G^{\mu\nu} = 0$, there would be no need of $j_1 = 0 = j_2 = g_1 = g_2$. Indeed, assuming $a = 0 = d$, let us examine the scalar equations of motion:

$$\begin{aligned} \partial_\mu\partial_\nu K_G^{\mu\nu} &= [1 + d(\square - m^2) + A_1\square + A_2\square^2] \partial^\mu\partial^\nu K_{\mu\nu} + (A_3 + A_4\square)\square K \\ &= P \partial^\mu\partial^\nu h_{\mu\nu} + Q \square h = 0 \quad . \end{aligned} \quad (24)$$

$$\begin{aligned} \square \eta_{\mu\nu} K_G^{\mu\nu} &= [1 + d(\square - m^2) + D A_3 + A_4\square] \square K + (D A_1 + A_2\square) \square \partial^\mu\partial^\nu K_{\mu\nu} \\ &= S \partial^\mu\partial^\nu h_{\mu\nu} + T \square h = 0 \quad . \end{aligned} \quad (25)$$

The equations (24) and (25) define P, Q, S, T unambiguously as functions of \square . In order to have only trivial solutions $\partial^\mu \partial^\nu h_{\mu\nu} = 0 = \square h$ the determinant $PT - QS$ must be a nonvanishing constant. Explicitly,

$$m^4(D-1)^2(b^2 - cf)\square^2 - (D-1)(2bm^2 + f(D-2))\square + D(D-1)m^2f + 1 = C \quad (26)$$

where $C = (PT - QS)/[m^4(D-1)]$ is required to be a non vanishing real constant thus leading to $\partial^\mu \partial^\nu h_{\mu\nu} = 0 = \square h$. Back in the scalar equation $\eta_{\mu\nu} K_G^{\mu\nu} = 0$ we deduce the null trace $h = 0$, thus guaranteeing $\hat{a}^2 K = 0$ on shell.

There are several solutions to (26). In particular, solutions I and II satisfy (26) with $C = 1$. Another interesting solution corresponds to constant coefficients:

$$b = -\frac{(D-2)f}{2m^2} \quad ; \quad c = \frac{(D-2)^2 f}{4m^4} \quad (\text{Solution III}) \quad (27)$$

It corresponds to $C = 1 + D(D-1)fm^2$ where f is an arbitrary real constant such that $C \neq 0$. The corresponding Lagrangian differs from the Fierz-Pauli theory by a square term:

$$\mathcal{L}^{III} = \mathcal{L}_{FP} + \frac{f}{8} [(D-2)R_L + 2(D-1)m^2 h]^2 \quad . \quad (28)$$

The fact that the equations of motion $\delta S^{III} = 0$ lead to $K^{\mu\nu} = 0$ directly follows from the on shell identity:

$$[(D-2)\partial_\mu \partial_\nu + 2m^2 \eta_{\mu\nu}] \frac{\delta S^{III}}{\delta h_{\mu\nu}} = m^2 [1 + fm^2 D(D-1)] [(D-2)R_L + 2(D-1)m^2 h] = 0 \quad (29)$$

Regarding a nonlinear completion for the third solution, we may have for instance

$$S_{III} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \left\{ R - \frac{m^2}{4} (h_{\mu\nu} h^{\mu\nu} - h^2) + \frac{f}{16} [(D-2)R + 2(D-1)m^2 h]^2 \right\} \quad (30)$$

where f is a real constant with inverse mass squared dimension such that $1 + D(D-1)fm^2 \neq 0$. Once again $h_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}^{(0)}$ and (28) is the quadratic truncation of (30) about flat space.

Remarkably, comparing with the usual massive theory (20), in S_{III} we have both an ultraviolet and an infrared modification. The non derivative mass term now departs from the usual FP structure but remains ghost free at quadratic level. In the usual FP case ($f = 0$) one can get rid of the Boulware-Deser ghost at nonlinear level by the addition of fine tuned non derivative terms [6]. Since the IR and UV modifications in (30) are interconnected, we believe that possible ghost free modifications of [6] must include both non derivative and higher order terms.

In order to find and classify solutions to the determinant condition (26) we should consider the coefficients b, f, c as Taylor series in \square and require that each power of \square in (26) vanishes. Although we are still struggling with the search of solutions to (26), we have been able so far to find a fourth class of solutions (Solution IV):

$$\begin{aligned}
f &= f_0 + f_1 \square \quad ; \quad b = b_0 + b_1 \square \quad ; \quad c = c_0 + c_1 \square \\
f_0 &= 0 \quad ; \quad b_0 = D \frac{f_1}{2} \quad ; \quad b_1 = \frac{f_1}{8 m^2} [D^2(D-1)f_1 m^4 - 4(D-2)] \quad ; \quad c_0 = D b_1 \quad ; \quad c_1 = \frac{b_1^2}{f_1} \quad (31)
\end{aligned}$$

The corresponding flat space Lagrangian is given by

$$\mathcal{L}^{IV} = \mathcal{L}_{FP} + \frac{f_1}{128} [N R_L + 8(D-1)m^2 h] \{ \square [N R_L + 8(D-1)m^2 h] + 8 D m^2 R_L \} . \quad (32)$$

where $N = D^2(D-1)m^4 f_1 + 4(D-2)$ is an arbitrary real number which follows from the arbitrariness of f_1 . Notice that f_1 is not just an overall factor since $N = N(f_1)$. An interesting subcase, we call solution IV-a, corresponds to choose f_1 such that $N = 0$. We end up with a second order theory which differs from the usual massive FP theory by a trivial field redefinition (Weyl transformation):

$$\mathcal{L}^{IV-a}[h_{\mu\nu}] = \mathcal{L}_{FP}[h_{\mu\nu}] + 2 \frac{(D-2)}{D} \partial^\mu h \partial^\alpha h_{\alpha\mu} - 2 \frac{(D-2)}{D^2} \partial^\mu h \partial_\mu h = \mathcal{L}_{FP}[h_{\mu\nu} - (2/D)\eta_{\mu\nu}h] . \quad (33)$$

The Weyl transformation $h_{\mu\nu} \rightarrow h_{\mu\nu} - (2/D)\eta_{\mu\nu}h$ is the only one which preserves the form of the FP mass term. Even though \mathcal{L}^{IV-a} differs trivially from the FP theory at quadratic level, a nonlinear completion of IV-a may be quite different from (20) with $c = 0$, for instance, we may have

$$S^{IV-a}[g_{\mu\nu}, g_{\mu\nu}^{(0)}] = \frac{1}{2 \kappa^2} \int d^D x \sqrt{-g} \left\{ R - \frac{m^2}{4} (h_{\mu\nu} h^{\mu\nu} - h^2) + \frac{(D-2)}{D^2} h [\square h - D \nabla^\mu \nabla^\nu h_{\mu\nu}] \right\} \quad (34)$$

Expanding S^{IV-a} around flat space ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) we recover (33) at quadratic level. According to [16], where a complete set of cubic and quartic (in powers of $h_{\mu\nu}$) vertices with two derivatives has been analyzed, there is apparently no hope of a ghost free (at nonlinear level) second order model.

In the general case $N \neq 0$ we may have the nonlinear fourth order completion:

$$S_{IV} = \frac{1}{2 \kappa^2} \int d^D x \sqrt{-g} \left\{ R - \frac{m^2}{4} (h_{\mu\nu} h^{\mu\nu} - h^2) + \frac{f_1}{264} \Psi (\square \Psi + 8 D m^2 R) \right\} . \quad (35)$$

where we have introduced the curved space scalar $\Psi = N R + 8(D-1)m^2 h$.

There are other Taylor series $f(\square)$, $b(\square)$, $c(\square)$ which solve the determinant condition (26). We leave for a future work the detailed analysis of those solutions altogether with a study of possible ghost-free models at nonlinear level.

3 The propagator

In order to confirm that the higher order (in derivatives) generalizations of the Fierz-Pauli theory suggested in the last section are indeed ghost and tachyon free, we examine here the

analytic structure of the propagator of those theories. In general, a Lagrangian of the form (6) leads to the propagator (suppressing indices)

$$G^{-1} = \frac{2 P_{SS}^{(2)}}{(\square - m^2) [d(\square - m^2) + 1]} - \frac{2 P_{SS}^{(1)}}{m^2 (2 + a m^2 \square - 2 d m^2)} + \frac{\left[A_{ww} P_{WW}^{(0)} + A_{ws} \left(P_{SW}^{(0)} + P_{WS}^{(0)} \right) + A_{ss} P_{SS}^{(0)} \right]}{K^{(0)}} \quad (36)$$

where the determinant $K^{(0)} = A_{ww}A_{ss} - A_{ws}^2$ is given in terms of the coefficients $A_{ij} = A_{ij}(\square)$ which are complicated functions of the coefficients a, b, c, d, f given in the appendix. The spin- s operators $P_{IJ}^{(s)}$ are also displayed in the appendix.

Now we can understand why we need to have $d = 0 = a$ from a different viewpoint. Namely, if $d \neq 0$ we would have a double pole in spin-2 sector which would necessarily lead to a spin-2 ghost. We need to chose $a = 0$ to avoid a massive vector particle. With $d = 0 = a$ the spin-2 and spin-1 sectors of the propagator (36) coincide with the corresponding sectors of the propagator of the usual second order FP theory. In the spin-0 sector it turns out that $K^{(0)} = -(PT - QS)/4 = -m^4(D - 1)C/4$ where C is defined by the left hand side of (26). Thus, the requirement of last section that C be a nonvanishing real constant is equivalent to the absence of poles in the spin-0 sector of the propagator. Therefore there is a simple interpretation of the Euler tensor approach in terms of the analytic structure of the propagator. Namely, it amounts to preserve the analytic structure of the usual FP propagator.

4 Euler tensor approach versus gauge embedment

Since there is another approach in the literature [25, 26, 27] where dual (higher order) models are generated by the addition of quadratic terms in the Euler tensor, it is worth pointing out the essential differences between those techniques.

As shown⁴ in [25], one can obtain in $D = 2 + 1$ the second order, Maxwell-Chern-Simons (MCS) theory of [28] via a Noether gauge embedment (NGE) of the first order self-dual (SD) model of [29]. Namely,

$$\mathcal{L}_{MCS} = \frac{1}{2} f_\mu K^\mu + \frac{1}{2 m^2} K_\mu K^\mu = \mathcal{L}_{SD} + \frac{1}{2 m^2} K_\mu K^\mu \quad . \quad (37)$$

Defining the transverse operator $E_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha} \partial^\alpha$, the Euler vector K_μ of the SD model is given by $K_\mu = \delta S_{SD} / \delta f^\mu = m E_{\mu\nu} f^\nu - m^2 f_\mu$. An arbitrary variation of the MCS action

$$\delta S_{MCS} = \int d^3x \left[K_\mu \delta f^\mu + \frac{K_\mu}{m^2} (m E^\mu{}_\nu \delta f^\nu - m^2 \delta f^\mu) \right] = \frac{1}{m} \int d^3x K^\mu E_{\mu\nu} \delta f^\nu \quad . \quad (38)$$

reveals the $U(1)$ symmetry $\delta f_\mu = \partial_\mu \phi$ of the MCS theory, contrary to the SD model which has no local symmetry.

⁴The first proof of equivalence between the Maxwell-Chern-Simons theory of [28] and the first order Self-dual model of [29] has appeared in [30], but for our purposes the work [25] is more convenient.

Moreover, (38) reveals also that the SD equations of motion $K_\mu = 0$ are embedded in the MCS ones $E_{\mu\nu}K^\nu = 0$. Similarly, the linearized new massive gravity (NMG) of [31], which is of fourth order in derivatives, can be obtained from the usual second order massive FP theory by adding quadratic terms in the FP Euler tensor, see [27]. The linearized NMG theory is invariant under $\delta h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu$ contrary to the FP theory which has no local symmetry. In both cases, SD/MCS and FP/NMG, we have a Noether gauge embedment.

Contrary to the Euler tensor approach of section 2, in the NGE it is not possible to prove that the new (higher order) equations of motion imply the old ones. In fact, this is not expected on general grounds since the old equations, like $K_\mu = 0$, are not gauge invariant. The on shell equivalence between the dual theories is more subtle in the NGE. For instance, in the SD/MCS case one defines a gauge invariant dual field $F_\mu = E_{\mu\nu}f^\nu/m$ and shows that the MCS equations of motion can be cast in the SD **form** $K_\mu = 0$ with $f_\mu \rightarrow F_\mu$. A similar dual map exists also in FP/NMG case, see [27].

There is a crucial difference also in the analytic structure of the propagator if we compare the Euler tensor approach of last section to the NGE. In the former, the changes in the propagator do not affect its analytic structure at all. The highest spin sectors remain unchanged while the contribution of the different spin-0 terms that we have introduced in the action have a vanishing net influence in $K^{(0)}$. On the other hand, in the NGE there appear new poles in the denominators even in the highest spin (physical) sector. However, when we saturate the two-point amplitude with sources satisfying the constraints imposed by the larger symmetries, the new poles have a vanishing residue, see [32, 33]. This already happens in the SD/MCS duality. In the MCS model there is a massless pole in the propagator, with vanishing residue, which is not present in the SD model. This is how the NGE solves the puzzle of having higher derivative theories while preserving unitarity⁵. It is a rather different solution as compared to the Euler tensor approach where the different higher order terms in the action are fine tuned in order to produce no contributions at all in the denominators in the propagator, thus avoiding extra poles from the start.

5 Conclusion

Here we have made embeddings of the Euler tensor as a technique to produce dual theories to the massive Fierz-Pauli model. The models obtained are in general of higher order in derivatives but remain unitary as demonstrated via analysis of the analytic structure of the corresponding propagators. The Euler tensor approach could be in principle generalized to other models. The coefficients appearing in the Ansatz (6) are in general analytic functions of $\square = \partial^\mu \partial_\mu$ which must satisfy condition (26). There are several solutions to (26). We have written down explicitly four set of solutions. In all cases the (usually) higher order equations of motion turn out to be equivalent to the second order Fierz-Pauli equations, thus leading to the Fierz-Pauli conditions and the Klein-Gordon equation. They describe a massive spin-2 particle if $D = 4$.

All models admit nonlinear completions in terms of two metrics, $g_{\mu\nu}$ and $g_{\mu\nu}^{(0)}$, see (20),(23),

⁵We thank Prof. José A. Helayel-Neto for a comment on that point.

(30) and (35). The solutions I and II are all of higher order in derivatives. The special subcase of solutions I, see (20), where $c(\square)$ is a constant has appeared before in [19] where, following [6], nonlinear non derivative terms have been added to render the model ghost free at nonlinear level. The authors of [19] have found interesting self-accelerating solutions.

For the solutions III and IV the situation is different. In solutions III the usual Fierz-Pauli (non derivative) mass term gets modified altogether with a higher order R^2 term and a second order modification of the Fierz-Pauli theory, see (23). So we believe that the addition of nonlinear non derivative terms must come together with nonlinear higher derivative terms in order to get rid of the Deser-Boulware ghost eventually. In the set of solutions IV once again the second order Fierz-Pauli terms must change along with higher order terms, thus requiring a more careful study of possible ghost free additions.

It is possible to lower the order (in derivatives) of (23) and (30) by introducing a scalar field. Since the use of a scalar field seems to be an important ingredient in the proof of absence of ghosts, at nonlinear level, in the higher order theories of [21], there is some hope of a generalization.

We are currently investigating the generalization of the Ansatz (6) to the nonsymmetric case, where we start with the nonsymmetric FP theory (1) and add quadratic terms in the nonsymmetric Euler tensor $\tilde{K}^{\mu\nu}$.

6 Appendix

Here we display the operators $P_{IJ}^{(s)}$ and the coefficients $A_{ij}(\square)$ mentioned in section 3.

Using, as building blocks the spin-0 and spin-1 projection operators acting on vector fields, respectively,

$$\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\square} \quad , \quad \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square} \quad , \quad (39)$$

we define the spin-s operators $P_{IJ}^{(s)}$ acting on symmetric rank-2 tensors in D dimensions:

$$\left(P_{SS}^{(2)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{2} \left(\theta^\lambda{}_\alpha \theta^\mu{}_\beta + \theta^\mu{}_\alpha \theta^\lambda{}_\beta \right) - \frac{\theta^{\lambda\mu} \theta_{\alpha\beta}}{D-1} \quad , \quad (40)$$

$$\left(P_{SS}^{(1)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{2} \left(\theta^\lambda{}_\alpha \omega^\mu{}_\beta + \theta^\mu{}_\alpha \omega^\lambda{}_\beta + \theta^\lambda{}_\beta \omega^\mu{}_\alpha + \theta^\mu{}_\beta \omega^\lambda{}_\alpha \right) \quad , \quad (41)$$

$$\left(P_{SS}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{3} \theta^{\lambda\mu} \theta_{\alpha\beta} \quad , \quad \left(P_{WW}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \omega^{\lambda\mu} \omega_{\alpha\beta} \quad , \quad (42)$$

$$\left(P_{SW}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{\sqrt{D-1}} \theta^{\lambda\mu} \omega_{\alpha\beta} \quad , \quad \left(P_{WS}^{(0)}\right)_{\alpha\beta}^{\lambda\mu} = \frac{1}{\sqrt{D-1}} \omega^{\lambda\mu} \theta_{\alpha\beta} \quad , \quad (43)$$

They satisfy the symmetric closure relation

$$\left[P_{SS}^{(2)} + P_{SS}^{(1)} + P_{SS}^{(0)} + P_{WW}^{(0)} \right]_{\mu\nu\alpha\beta} = \frac{\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}}{2} \quad . \quad (44)$$

The coefficients $A_{ij}(\square)$ are given by

$$A_{ss} = z + y - (c_1 + c_2)\square + c_3\square^2 \quad , \quad A_{ww} = z + y(D-1) \quad , \quad A_{ws} = \sqrt{D-1}(c_2\square/2 - y) \quad . \quad (45)$$

where

$$c_1 = 1 + a\frac{m^4}{2} + d(\square - 2m^2) \quad , \quad (46)$$

$$c_2 = -1 + d(D-2)(\square - m^2) + [d + 2m^2b(D-2) + cm^4 + f(D-2)^2]\square - m^2(D-1)[bm^2 + f(D-2)] - am^4 \quad . \quad (47)$$

$$c_3 = \frac{dD}{2} + m^2b(D-2) + \frac{cm^4}{2} + \frac{f}{2}(D-2)^2 \quad . \quad (48)$$

$$z = \frac{(\square - m^2)}{2} [d(\square - m^2) + 1] \quad . \quad (49)$$

$$\begin{aligned} y = & -\frac{(\square - m^2)}{2} + d\square \left[\frac{(D-1)}{2}\square + m^2(3-D) \right] \\ & + m^2 \left[b(D-2) + c\frac{m^2}{2} + \frac{f(D-2)^2}{2m^2} \right] \square^2 + \frac{f}{2}m^4(D-1)^2 \\ & - m^2(D-1) [bm^2 + f(D-2)] \square \quad . \end{aligned} \quad (50)$$

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